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CERTAIN QUALITATIVE INVESTIGATION METHODS FOR DYNAMIC SYSTEMS CONNECTED WITH FIELD ROTATION

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We show that the inevitability of realizing bifurcations connected with a double (and triple) limit cycle or with a separatrix loop can, in some cases, be detected from the global evaluation of trajectory behavior under parameter variations by reckoning the sign of the saddle term, and we turn our attention to new possibilities of tracing the bifurcations arising from the use of a monotonic field rotation. The methods of bifurcation theory [1] are widely used in problems of mechanics. However, in the general case, the problem of investigating all possible bifurcations is difficult and regular methods for solving it do not exist. There are no criteria locally connected with the points of the phase space or of the parameter space, which stipulate the actual realizability in a concrete dynamic system of bifurcations connected with a separatrix loop. A similar situation exists for bifurcations connected with the arising of limit cycles from the condensation of trajectories, because we usually know neither the equations of the limit cycle nor the parameter values under which it arises. In a number of the methods used in bifurcation theory an important role is played by the local rotation of the field in a neighborhood of the singular trajectories of the system [1, 2]. In a number of cases the carrying over of this idea to the whole phase space and to the parameter space in the large (realizable in the presence of specific singularities of the system being investigated) permits us not only to trace all bifurcations possible in the system but also to predetermine the disposition of the bifurcation curves or surfaces.

1. We examine the system, considered in [3] by the small parameter method (the phase space is a cylinder)

$$\varphi^{\cdot} = y, \quad y^{\cdot} = \beta - \sin\varphi - \lambda y - 2\alpha s\,\frac{y}{s^2 + y^2} \tag{1.1}$$

By the known methods of qualitative theory we can detect that there are two equilibrium states on the axis $y = 0$ for all values of the parameters $\alpha > 0$, $s > 0$, $\lambda > 0$, $0 < \beta < 1$: O_1 (arc sin β, 0), a stable node or focus and O_2 (π — arc sin β, 0), a saddle. On the lower half-cylinder the trajectories go from infinity onto the upper half-cylinder. There are no cycles [4, 5] on the lower half-cylinder and around the point O_1. All bifurcations can occur only on the upper half-cylinder.

For large λ the structure of the phase space partitioning is uniquely determined in comparison with the system

$$\varphi^{\cdot} = y, \quad y^{\cdot} = \beta - \sin\varphi - 2\varkappa y \tag{1.2}$$

As is known [4, 6], for each β $(0 < \beta < 1)$ there exists $\varkappa^*(\beta)$ such that when $\varkappa > \varkappa^*$ the ω-separatrix of the saddle of system (1.2), going onto the upper half-cylinder, does not intersect $y = 0$ and goes to infinity on the upper half-cylinder. If $\lambda > 2\varkappa$, the direction field of (1.1) turns clockwise relative to the direction field of (1.2). Therefore, if $\varkappa > \varkappa^*$ and $\lambda > 2\varkappa$, the ω-separatrix of the saddle of system (1.1) also should go to infinity. There are no cycles. The phase space partitioning structure is equivalent to that shown in Fig. 1(0).

Let us observe the variation of the qualitative structure and the possible bifurcations for fixed β $(0 < \beta < 1)$ and $s > 0$ in the $\alpha\lambda$ parameter plane. The qualitative structure does not depend upon the β and s chosen. The qualitative structures realized along the straight line $\lambda = 0$ are known [5]. There exist α_1 and α_2 such that a partitioning structure without limit cycles is realized on a piece $0 < \alpha < \alpha_1(\beta, s)$ of the α-axis. A structure with two limit cycles (a stable lower one and an unstable upper one) is realized in the upper half-cylinder on the piece $\alpha_1(\beta, s) < \alpha < \alpha_2(\beta, s)$. A structure with one unstable limit cycle is realized on the piece $\alpha_2(\beta, s) < \alpha < \infty$. A structure with a double limit cycle arising from a condensation of trajectories corresponds to the point $\alpha = \alpha_1(\beta, s)$. A structure with a separatrix loop covering the upper half-cylinder corresponds to the point $\alpha = \alpha_2(\beta, s)$.

Let us observe the changes in the qualitative structures and in the possible bifurcations as λ increases along the straight lines $\alpha = \alpha_0$. We examine three cases.

1°. $\alpha_0 > \alpha_2$. As λ increases from the value $\lambda = 0$, the term $-\lambda y$ appears in the equation and the point at infinity becomes unstable. Stable limit cycle emerges from infinity. This structure is shown on Fig. 1(2). There are two limit cycles on the upper half-cylinder. As λ increases the direction field rotates clockwise and the limit cycles come together monotonically (the stable one drops and the unstable one rises). Since the partitioning structure shown in Fig. 1(0) (no cycles now) is automatically realized when $\lambda > \varkappa^*$, there exists $\lambda = \lambda^{++}(\alpha; \beta, s)$ for which the limit cycles merge, forming a double semistable limit cycle. As λ increases from the bifurcation value λ^{++} the double limit cycle disappears.

2°. $\alpha_1 < \alpha_0 < \alpha_2$. As λ increases from the value $\lambda = 0$, a third (stable) limit cycle emerges from infinity. This structure is shown on Fig. 1(3). As λ increases the upper and lower stable limit cycles drop monotonically, while the unstable one located between them rises monotonically. Since for $\lambda > \varkappa^*$ cycles are absent and with increasing λ the field turns monotonically in a clockwise direction, there exist $\lambda = \lambda^{++}(\alpha; \beta, s)$, which corresponds to the merging of the unstable limit cycle with the stable upper one and, also, $\lambda = \lambda_0(\alpha; \beta, s)$, which corresponds to the confluence of the lower stable cycle with the separatrix loop at the upper half-cyclinder (a loop can only arise as the result of contraction of a stable limit cycle into a loop, since the saddle quantity $(P_x' + Q_y') = -(2\alpha/s + \lambda)$ is negative [1, 2]).

3°. $0 < \alpha_0 < \alpha_1$. As λ increases from the value $\lambda = 0$, a stable limit cycle emerges from infinity, which drops monotonically with increasing λ. Since for $\lambda > \varkappa^*$ cycles are absent and the saddle quantity is negative, there exists $\lambda = \lambda_0(\alpha; \beta, s)$ which corresponds to the loop of the saddle separatrix at the upper half-cylinder. For $\lambda = \lambda_0$ the stable limit cycle merges with the loop of the separatrix. If the monotonic rotation of the field does not everywhere increases the pitch (the distance between coils) of the spiral covering the cylinder, then the further possibility remains of a double limit cycle arising from the condensation of trajectories with a subsequent separation of the double limit cycle into simple ones — a stable and an unstable. Such a possibility is realized as λ increases along the straight line $\alpha = \alpha_0 < \alpha_1$ if α_0 is sufficiently close to α_1.

A phase space partitioning structure with a double semistable limit cycle on the upper half-cylinder corresponds to the point $\lambda = 0$, $\alpha = \alpha_1$. Since the direction field rotates in opposite senses as λ increases and as α decreases (clockwise and counter-clockwise, respectively), the limit cycle separates into two as λ increases and vanishes as α decreases. Considerations of continuity imply that in the $\alpha\lambda$-plane a bifurcational curve $\lambda = \lambda^+ (\alpha;\ \beta,\ s)$ emerges from point $\lambda = 0$, $\alpha = \alpha_1$ with a negative slope, whose binary cycle does not collapse. The straight line $\alpha = \alpha_0 < \alpha_1$ intersects this curve if α_0 is sufficiently close to α_1.

Let us observe the variation of the qualitative structures as λ increases along the straight line $\alpha = \alpha_0 < \alpha_1$ when α_0 is sufficiently close to α_1. The structure of Fig. 1(0) is realized when $\lambda = 0$. There are no cycles. As we pass to positive λ a stable limit cycle emerges from infinity, which drops as λ increases. For the value $\lambda = \lambda^+ (\alpha_0;\ \beta,\ s)$ a double limit cycle appears below the stable limit cycle (the double limit cycle cannot arise above the stable limit cycle which emerged from infinity because above the cycle the pitch of the spiral winding onto the stable cycle from above can only increase as the field rotates clockwise). With a further increase of λ the double limit cycle separates into a lower stable one and an upper unstable one, and the partitioning structure of Fig. 1(3) is realized. Under an additional increase in λ the stable cycle drops and the unstable one rises. Since there are no cycles for $\lambda > \varkappa^*$, two more bifurcations are necessarily realized in the interval $\lambda^+ < \lambda < \varkappa^*$: the fusion of the stable and the unstable limit cycles on the bifurcation curve $\lambda = \lambda^{++} (\alpha_0;\ \beta,\ s)$ and the arising of a separatrix loop on the bifurcation curve $\lambda = \lambda_0 (\alpha_0;\ \beta,\ s)$ as the stable (since the saddle term is negative) limit cycle contracts to it with increase in λ.

2. We trace the disposition of the bifurcation curves in the $\alpha\lambda$-plane. The bifurcation curve $\lambda = \lambda^{++} (\alpha;\ \beta,\ s)$ exists for all values $\alpha > \alpha_1$ and for values $\alpha < \alpha_1$ sufficiently close to α_1. The curve $\lambda = \lambda^{++} (\alpha;\ \beta,\ s)$ has a negative slope. The latter follows from the fact that on a curve with a positive slope the vector field rotates monotonically under a simultaneous increase or decrease of the parameters α and λ, and a double limit cycle could not exist in such case. The bifurcation curve $\lambda = \lambda^+ (\alpha;\ \beta,\ s)$ starts at the point $\lambda = 0$, $\alpha = \alpha_1$, exists in a certain neighborhood of this point to the left, and, for the same reason as for the curve λ^{++}, has a negative slope.

The curve $P_x' + Q_y' = 0$ does not have real branches in the phase space if $\alpha < 4s\lambda$. Therefore, for the condition $\alpha < 4s\lambda$ there cannot be more than one cycle girding the phase cylinder [7]. This circumstance helps us trace the behavior of the curves λ^+ and λ^{++}. As α decreases the curves λ^+ and λ^{++} can go neither to infinity (since they cannot intersect the straight line $\lambda = \varkappa^*$) nor to the λ-axis (since they cannot intersect the straight line $\alpha = 4s\lambda$) nor to the α-axis (since they have negative slopes). As α decreases the curves λ^+ and λ^{++} can end up only at the corner point corresponding to the joining of the bifurcation curves λ^+ and λ^{++}. The system has a triple limit cycle for values of the parameters corresponding to this corner point.

The bifurcation curve $\lambda = \lambda_0 (\alpha;\ \beta,\ s)$ exists on the interval $0 \leqslant \alpha \leqslant \alpha_2$. Any straight line $\alpha = \alpha_0$ $(0 \leqslant \alpha_0 \leqslant \alpha_2)$ or $\lambda = \lambda_1$ $(0 \leqslant \lambda_1 \leqslant \varkappa^*)$ intersects it only once because the direction field turns monotonically as λ or α increase. It passes

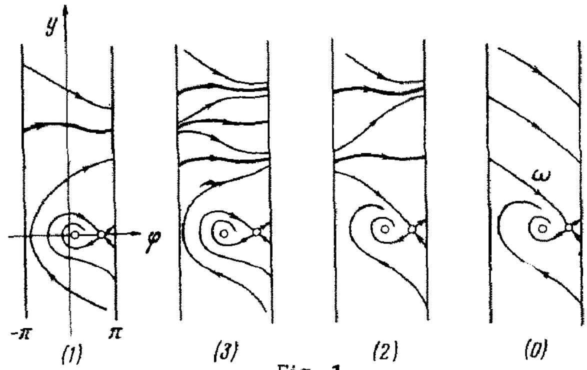

Fig. 1

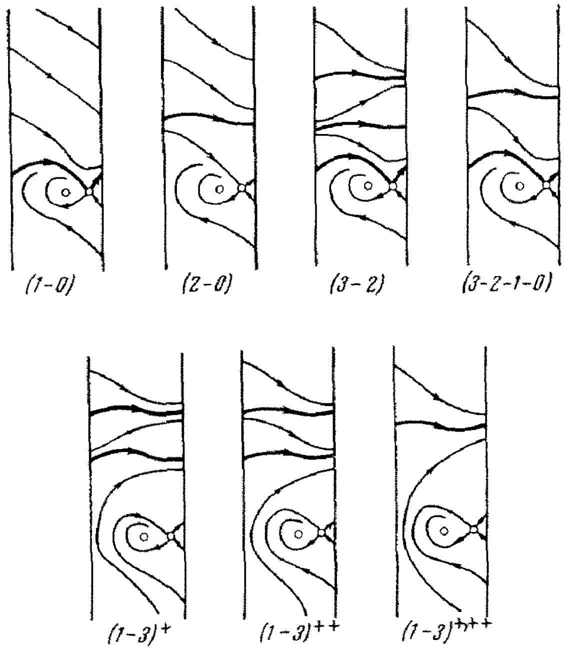

Fig. 2

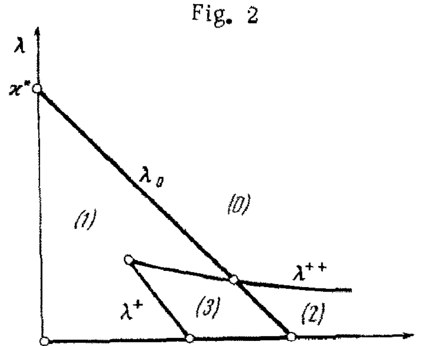

Fig. 3

through the point $\alpha = 0$, $\lambda = \varkappa^*$ (by definition of $\varkappa^*$ [4, 6]) and through the point $\alpha = \alpha_2$, $\lambda = 0$ (by definition of α_2 [5]). For the same reason as for the curves λ^+ and λ^{++} the curve λ_0 has a negative slope.

Note. Structures on the straight line $\lambda = 0$ are known only to within an additional even number of cycles girding the cylinder [5]; therefore, there remains the inevitable logical possibility of the existence of "doubles" of bifurcation curves λ^+ and λ^{++}.

Let us ascertain the disposition of curve λ_0 relative to curves λ^+ and λ^{++}, assuming the absence of "doubles". If, with increasing λ, a separatrix loop arises around the cylinder and then collapses before the appearance of a double limit cycle, then the phase space partitions into trajectories without limit cycles; the ω-separatrix of the saddle winds around the upper half-cylinder, going to infinity. For a subsequent increase in λ limit cycles can no longer arise because the field rotates clockwise and the spiral's pitch on the upper half-cylinder increases. No part of curve λ_0 can be located below curve λ^+. Therefore, curves λ_0 and λ^+ cannot intersect. The curve λ_0 also cannot pass through the corner point of the joining of curves λ^+ and λ^{++}. To such a point there must correspond a phase space partitioning structure with a triple stable limit cycle and a simple stable separatrix loop on the upper half-cylinder (the saddle term is nonzero and is negative). The presence of these elements in the phase space partitioning structure is possible only when there exists an unstable limit cycle separating them. The assumption of the possibility of such a structure at the corner point leads to a contradiction with the assumption that this is a corner point (the rotation of the field as α decreases can transfer such a structure into a structure with one limit cycle, which realizes to the left of the straight line $\alpha = 4s\lambda$ by a passage through the bifurcation curve λ^+, but this is impossible if the initial point is a corner point).

The curve λ_0 intersects λ^{++} to the right of the corner point. The partitioning of the parameter space for $\beta = \text{const}$ $(0 < \beta < 1)$ and $s = \text{const}$ is shown in Fig. 3. The digits $0-3$ mark regions in the parameter space, corresponding to the structurally stable spaces in Fig. 1 marked by those same digits. The digits indicate the number of cycles. The bifurcation curves in Fig. 3, separating the corresponding regions, correspond to the structurally unstable spaces in Fig. 2, marked by two or four digits. The signs + and ++ in Fig. 2 indicate, respectively, the belonging to the bifurcation curves λ^+ and λ^{++}. The limit cycle in Fig. $2(1-3)^{+,++}$ is a threefold one.

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CASE OF A GENERATING FAMILY OF QUASI-PERIODIC SOLUTIONS IN THE THEORY OF SMALL PARAMETER

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We consider the problem of the existence and the stability in-the-small of periodic solutions of systems of ordinary differential equations with a small parameter μ, which in the generating approximation ($\mu = 0$) admit of a family of quasi-periodic solutions (we are concerned only with the solutions belonging to the indicated family when $\mu = 0$). The case to be investigated is in a specific sense a more general case of the unisolated generating solution in the small parameter theory and, therefore, includes everything previously treated by Malkin [1], Blekhman [2], and others. The main difficulty in the investigation is the presence of a multiple zero root in the characteristic determinant of the problem's generating system, to which both simple as well as quadratic elementary divisors [3] correspond. This fact predestines the presence of three groups of stability criteria for the solution being examined. The method for constructing these criteria, proposed here, assumes, in contrast to a previous one [1], the preliminary determination of not only the generating approximation but also the first one to the desired periodic solution. Particular aspects of the general "mixed" problem treated here were studied earlier in [4, 5].

1. Existence of a periodic solution. At present relatively general integrability tests and integration methods for systems of high-order nonlinear differential equations have been worked out only for autonomous canonical systems [6]. The successive use of these methods leads, in the case of a sign-definite Hamiltonian function, to the determination of a general quasi-periodic integral. The conjugate canonic variables of the problem are here expressed as 2π-periodic functions of the quantities

$$\psi_s = \nu_s t + \alpha_s \tag{1.1}$$

and also of the mutually independent integration constants h_s ($s = 1, 2, \ldots$). Naturally, the total number of quantities ψ_s, h_s equals the order of the original system. If the quantities α_s are integration constants, also independent of each other and of h_s (and we assume this is so in what follows), then the quantities ψ_s acquire the nature of partial rapidly-rotating phases and, moreover, the partial frequencies ν_s depend, as does